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NONLINEAR ANALYSIS OF REISSNER PLATES ON AN ELASTIC FOUNDATION BY THE BEM

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Abstract—The fundamental solutions and the boundary element method for obtaining numerical solutions of nonlinear Reissner plates on an elastic foundation are presented in the paper. The derivation of the fundamental solutions is mainly based on two of Hu's functions and the resolution method of differential operator. An incremental form of the boundary integral equations is suggested to achieve linearization of the original nonlinear equations. The plate may be moderately thick or sandwich plates with (or without) elastic foundation. Finally, three examples are considered to illustrate the correctness and accuracy of the proposed method.

NOMENCLATURE

C	$5Et/12(1+\nu)$, for a homogeneous plate; $G_c(h+t)$, for a sandwich plate
C_w	a part of boundary $\partial\Omega$ of the solution domain Ω, on which deflection w is prescribed; C_{M_n}, C_R, etc. are defined similarly
D	$Et^3/12(1-\nu^2)$ for a homogeneous plate; $E(h+t)^2t/2(1-\nu^2)$ for a sandwich plate
E	modulus of elasticity
G	$E/2(1+\nu)$
G_c	core shear modulus
G_f	shear modulus of Pasternak-type foundation
h	core thickness
k	reaction coefficient of Winkler-type foundation
k_f	reaction coefficient of Pasternak-type foundation
M_{ii}	bending moment
M_{ij}	twisting moment ($i \neq j$)
N_{ij}	membrane force tensor
n_i	components of the outward normal to the boundary $\partial\Omega$
q	lateral distributed load
Q_i	transverse shear force
r	$(x^2+y^2)^{1/2}$
R_n	$Q_in_i+N_nw_{,n}+N_{ns}w_{,s}$
s_i	components of the tangent to the boundary $\partial\Omega$
t	plate thickness (or face-sheet thickness)
u_1, u_2	in-plane displacements
w	lateral deflection
δ	variational symbol
δ_{ij}	the Kronecker delta
θ	arctg (y/x)
λ	$\sqrt{10}/t$ for a homogeneous plate; $4(1+\nu)G_c/E(h+t)t$ for a sandwich plate
ν	Poisson's ratio
∇^2	$\partial^2/\partial x^2+\partial^2/\partial y^2$
ψ_i	the average rotations normal to the plate mid-surface
$(\bar{\ })$	over a symbol denotes prescribed value.

1. INTRODUCTION

The boundary element method (BEM) is very popular as a numerical method in computational mechanics. The method has been widely used in linear bending problems (small deflection) of thin plates [see reference lists in Tottenham (1979) and Stern and Lin (1986)] and moderately thick plates [see e.g. Weeen (1982), Karam and Telles (1988) and Wang *et al.* (1992)]. As a further progressive step, various boundary integral formulations have been developed too treat large deflection of plates in the decade. Among the early proposals for analysing finite deflection of thin plates as the so-called direct formulation of Kamiya and Sawaki (1982a,b). Consequently, they extended their procedure to the case of sandwich plates and shells [see e.g. Kamiya *et al.* (1983) and Kamiya and Sawaki (1984, 1986)].

Another approach was proposed by Tanaka (1984). He obtained a couple boundary and inner domain integral equations in terms of stress and displacement functions. Later on, many researchers investigated the BEM for large deflection of plates with the so-called generalized Green identities [see e.g. Ye and Liu (1985), Ye (1991a) and Wang *et al.* (1991)], the dual reciprocity process (Sawaki *et al.*, 1989), the weighted residual method (Lei *et al.*, 1990) and the spline function (Ye, 1991b). Postbuckling problems of thin plates have been examined by Qin and Huang (1990) and Huang and Qin (1990) by using a newly derived fundamental solution. Geometrically nonlinear plates on elastic foundation have been considered by Katsikadelis (1991). Most of the developments in the field can also be found in Beskos' work (1991).

More recently, Jin and Qin (1993) have developed a set of boundary integral equations for analysing large deflection of Reissner plates based on the variational approach as well as a modified variational functional. So far, however, there are very few results by BEM for nonlinear Reissner plates on elastic foundations.

In this paper, a set of boundary integral formulations for nonlinear plates on an elastic foundation is established by the variational approach (Jin and Qin, 1993). The plate may have arbitrary shape and its boundary may be subjected to any type of boundary conditions. Specifically, we derive, as the most important step of the BEM application, a group of fundamental solutions for a Reissner plate on an elastic foundation by means of the resolution method of differential operator and two of Hu's functions (Hu, 1963). An iterative scheme is suggested to calculate domain unknown variables. Three examples of a square plate, a circular plate and a 60° skew sandwich plate are numerically studied to illustrate the efficiency and accuracy of the present approach.

2. BASIC EQUATIONS AND THEIR FUNDAMENTAL SOLUTIONS

2.1. *Basic equations*

Consider a Reissner plate of uniform thickness t, occupying a two-dimensional arbitrary shaped region Ω bounded by its boundary $\partial\Omega$ and resting on an elastic foundation. We use a cartesian coordinate system in which the x- and y-axes lie in the plate middle plane. Throughout this paper, repeated indices imply the summation convention of Einstein. The indices i, j and k take values in the range $\{1, 2\}$, and m takes a value in the range $\{3, 4, 5\}$. The nonlinear behaviour of the plate for moderately large deflections is, in this case, governed by the following equations (Lei *et al.*, 1990):

(i) Equilibrium equations in Ω

$$N_{ij,j} = 0, \quad (i = 1, 2) \tag{1a, b}$$

$$M_{ij,j} - Q_i = 0, \quad (i = 1, 2) \tag{1c, d}$$

$$Q_{i,i} + N_{ij} w_{,ij} - q + \bar{k} w = 0. \tag{1e}$$

(ii) Constitutive relationships in Ω

$$N_{ij} = N^n_{ij} + N^l_{ij} \tag{2a}$$

$$N^l_{ij} = Gt\left\{u_{i,j} + u_{j,i} + \frac{2\nu}{1-\nu} u_{k,k}\delta_{ij}\right\}, \tag{2b}$$

$$N^n_{ij} = Gt\left\{w_{,i} w_{,j} + \frac{\nu}{1-\nu} w_{,k} w_{,k}\delta_{ij}\right\}, \tag{2c}$$

$$M_{ij} = \frac{1-\nu}{2} D\left(\psi_{i,j} + \psi_{j,i} + \frac{2\nu}{1-\nu}\psi_{k,k}\delta_{ij}\right), \tag{2d}$$

$$Q_i = C(w_{,i} - \psi_i). \tag{2e}$$

(iii) Natural boundary conditions

$$N_n = N_{ij}n_in_j = \bar{N}_n \,(\text{on } C_{N_n}), \qquad N_{ns} = N_{ij}n_is_j = \bar{N}_{ns}\,(\text{on } C_{N_{ns}}), \tag{3a, b}$$

$$M_n = M_{ij}n_in_j = \bar{M}_n \,(\text{on } C_{M_n}), \qquad M_{ns} = M_{ij}n_is_j = \bar{M}_{ns}\,(\text{on } C_{M_{ns}}), \tag{3c, d}$$

$$R_n = Q_in_i + N_nw_{,n} + N_{ns}w_{,s} = \bar{R}_n \,(\text{on } C_R). \tag{3e}$$

(iv) Essential boundary conditions

$$u_n = u_in_i = \bar{u}_n \,(\text{on } C_{u_n}), \qquad u_s = u_is_i = \bar{u}_s \,(\text{on } C_{u_s}), \tag{4a, b}$$

$$\psi_n = \psi_in_i = \bar{\psi}_n \,(\text{on } C_{\psi_n}), \qquad \psi_s = \psi_is_i = \bar{\psi}_s \,(\text{on } C_{\psi_s}), \tag{4c, d}$$

$$w = \bar{w}\,(\text{on } C_w), \tag{4e}$$

$$(\partial\Omega = C_{u_n} \cup C_{N_n} = C_{u_s} \cup C_{N_{ns}} = C_{\psi_n} \cup C_{M_n} = C_{\psi_s} \cup C_{M_{ns}} = C_w \cup C_R),$$

where a comma followed by a subscript indicates partial differentiation with respect to that subscript, and the other untold symbols are listed in the Nomenclature.

2.2. *Fundamental solutions*

The fundamental solutions play an important role in the derivation of the boundary integral equation. In this subsection, the construction of the fundamental solutions for Reissner plates on an elastic foundation will be discussed in detail. The foundation may be Winkler-type or Pasternak-type (Kerr, 1964). The governing equations used for deriving the fundamental solutions are:

(i) Equations corresponding to in-plane deformations (Lei *et al.*, 1990):

$$N^l_{ij,j} = 0 \quad (i = 1, 2). \tag{5a, b}$$

(ii) Equations corresponding to bending deformations in this case (Petrolito, 1989):

$$D[\psi_{x,xx} + 0.5(1-\nu)\psi_{x,yy} + 0.5(1+\nu)\psi_{y,xy}] + C(w_{,x} - \psi_x) = 0, \tag{6a}$$

$$D[0.5(1+\nu)\psi_{x,xy} + 0.5(1-\nu)\psi_{y,xx} + \psi_{y,yy}] + C(w_{,y} - \psi_y) = 0, \tag{6b}$$

$$C(\nabla^2 w - \psi_{x,x} - \psi_{y,y}) + \bar{k}w - q = 0, \tag{6c}$$

where $\bar{k}$ is the subgrade reaction operator, $\bar{k} = k$ for Winkler-type foundation, $\bar{k} = k_f - G_f\nabla^2$ for Pasternak-type foundation (Kerr, 1964).

The fundamental solution corresponding to eqns (5a, b) is obviously Kelvin's solution and can be found in the paper of Qin and Huang (1990).

In the following attention will be focused on finding the fundamental solution of eqns (6a–c).

The coupling of eqns (6a–c) makes it difficult to generate the fundamental solutions. To by-pass this problem, two of Hu's functions, g and f, are introduced such that

$$\psi_x = g_{,x} + f_{,y}, \quad \psi_y = g_{,y} - f_{,x}. \tag{7a, b}$$

The expressions (7a, b) are always possible but the solution is not unique. Indeed

$$g_{0,x} + f_{0,y} = 0, \quad g_{0,y} - f_{0,x} = 0 \tag{8a, b}$$

are Cauchy–Riemann equations the solution of which always exists. As a consequence, ψ_x

and ψ_y remain unchanged if f and g are replaced by $f+f_0$ and $g+g_0$. This important property will be used in the subsequent part of the paper. The solution of eqns (8a, b) may conveniently be expressed in a complex variable form:

$$f_0+ig_0 = \phi(x+iy) \tag{9}$$

where $i = \sqrt{(-1)}$.

The substitution of eqns (7a, b) into eqns (6a, b), leads to

$$\partial[D\nabla^2 g+C(w-g)]/\partial x+\partial[D(1-\nu)\nabla^2 f/2-Cf]/\partial y = 0, \tag{10a}$$

$$\partial[D\nabla^2 g+C(w-g)]/\partial y-\partial[D(1-\nu)\nabla^2 f/2-Cf]/\partial x = 0. \tag{10b}$$

If the contents of the two brackets are considered as two independent generalized functions, eqns (10a, b) are of the same form as eqns (8a, b). Therefore eqns (10a, b) also represent a set of Cauchy–Riemann equations and, in the same manner as eqn (9), we can set

$$[D(1-\nu)\nabla^2 f/2-Cf]+i[D\nabla^2 g+C(w-g)] = F(x+iy). \tag{10c}$$

Equation (10c) is a nonhomogeneous equation with independent variables f, g and w. Its solution can be composed of a homogeneous solution part and a particular part. Since $F(x+iy)$ is a harmonic function, the particular solutions of eqn (10c) can be taken in the form

$$f_1+ig_1 = -F(x+iy)/C \quad \text{and} \quad w_1 = 0. \tag{10d}$$

It is obvious that the particular solution (10d) leads to vanishing deflection and rotations (i.e. $w = \psi_x = \psi_y = 0$). Therefore the particular solutions may be omitted and we only need to consider the homogeneous part of eqn (10c):

$$D(1-\nu)\nabla^2 f/2-Cf = 0, \tag{11a}$$

$$D\nabla^2 g+C(w-g) = 0. \tag{11b}$$

The substitution of eqns (7a, b) and (11b) into eqn (6c), leads to

$$D\nabla^4 g+\frac{\bar{k}}{C}D\nabla^2 g-\bar{k}g+q = 0. \tag{12a}$$

As a result, we obtain, for f and g, the following set of differential equations:

$$D\nabla^4 g+\frac{\bar{k}}{C}D\nabla^2 g-\bar{k}g+q = 0, \tag{12b}$$

$$\nabla^2 f-\lambda^2 f = 0. \tag{12c}$$

The corresponding displacements and rotations are obtained from the following relations:

$$w = g-D\nabla^2 g/C, \tag{12d}$$

$$\psi_x = g_{,x}+f_{,y}, \tag{7a}$$

$$\psi_y = g_{,y}-f_{,x}. \tag{7b}$$

Equation (12c) is the well-known modified Helmholtz equation and its fundamental solution is

$$f^*(r) = \frac{\lambda^2}{2\pi C}\mathrm{K}_0(\lambda r), \tag{13}$$

where $\mathrm{K}_0(\ )$ is a modified Bessel function of zero order of the second kind.

The next step is to derive the fundamental solution of eqn (12a). To this end, consider the homogeneous equation

$$D\nabla^4 g^* + \frac{\bar{k}}{C}D\nabla^2 g^* - \bar{k}g^* = D(\nabla^2 + C_1)(\nabla^2 - C_2)g^* = \delta(P, Q), \tag{14}$$

in which $\delta(P, Q)$ is the Dirac delta function, P and Q denote the source point and the field point, respectively, and

$$C_1 = k/2C\sqrt{(k/2C)^2 + k/D}, \quad C_2 = \sqrt{(k/2C)^2 + k/D} - k/2C$$

for Winkler-type foundation, or

$$C_1 = (\sqrt{b} + k_f/C + G_f/D)/2(1 - G_f/C), \quad C_2 = (\sqrt{b} - k_f/C - G_f/D)/2(1 - G_f/C),$$
$$b = (k_f/C + G_f/D)^2 + 4k_f(1 - G_f/C)/D$$

for a Pasternak-type foundation.

To find the solution of eqn (14), we set

$$(\nabla^2 - C_2)g^* = A. \tag{15}$$

It follows from eqn (14) that

$$D(\nabla^2 + C_1)A = \delta(P, Q). \tag{16}$$

The solution of eqn (16) can be easily obtained as

$$A = Y_0(\sqrt{C_1}r)/4D \tag{17}$$

in which $Y_0(\ )$ is the Bessel function of zero order of the second kind.

In a similar way, let

$$(\nabla^2 + C_1)g^* = B, \tag{18}$$

then we have

$$D(\nabla^2 - C_2)B = \delta(P, Q). \tag{19}$$

The fundamental solution of eqn (19) is easily found to be

$$B = \frac{1}{2\pi D}\mathrm{K}_0(\sqrt{C_2}r). \tag{20}$$

Subtracting eqn (15) from eqn (18) and by using eqns (17) and (20), the fundamental solution g^* can be given in the form

$$g^* = \frac{B - A}{C_1 + C_2} = \frac{1}{C_1 + C_2}\left[\frac{1}{2\pi D}\mathrm{K}_0(\sqrt{C_2}r) - Y_0(\sqrt{C_1}r)/4D\right]. \tag{21}$$

In the absence of elastic foundation ($C_1 = C_2 = 0$), the solution (21) reduces to

$$g^* = \frac{1}{8\pi D} r^2 \ln r. \tag{22}$$

Substituting eqns (13) and (21) [or (22)] into eqns (7a, b) and (12d), we have

$$w^* = g^* - D\nabla^2 g^*/C, \tag{23a}$$

$$\psi_x^* = \partial g^*/\partial x + \partial f^*/\partial y, \tag{23b}$$

$$\psi_y^* = \partial g^*/\partial y - \partial f^*/\partial x. \tag{23c}$$

3. BOUNDARY INTEGRAL FORMULATION

The boundary integral equation for nonlinear Reissner plates on an elastic foundation can be established by the variational approach (Jin and Qin, 1993). The approach is mainly based on a modified variational principle. Following the line of argument of Jin and Qin (1993), the modified principle for the present problem can be stated as

$$\delta\Pi^m = 0 \Rightarrow \text{eqns (1), (3) and (4)}, \tag{24}$$

where

$$\Pi^m = \Pi_1 + \int_{C_{u_n}} (\bar{u}_n - u_n) N_n \, dc + \int_{C_{u_s}} (\bar{u}_s - u_s) N_{ns} \, dc + \int_{C_{\psi_n}} (\bar{\psi}_n - \psi_n) M_n \, dc + \int_{C_{\psi_s}} (\bar{\psi}_s - \psi_s) M_{ns} \, dc + \int_{C_w} (\bar{w} - w) R_n \, dc, \tag{25}$$

$$\Pi_1 = \iint_\Omega (U - wq) \, d\Omega - \int_{C_{N_n}} \bar{N}_n u_n \, dc - \int_{C_{N_{ns}}} \bar{N}_{ns} u_s \, dc - \int_{C_{M_n}} \bar{M}_n \psi_n \, dc - \int_{C_{M_{ns}}} \bar{M}_{ns} \psi_s \, dc - \int_{C_R} \bar{R}_n w \, dc, \tag{26}$$

$$2U = N_{ij}(u_{i,j} + u_{j,i} + w_{,i} w_{,j})/2 + M_{ij}(\psi_{i,j} + \psi_{j,i})/2 + Q_i(w_{,i} - \psi_i) + kw^2.$$

The terminology "modified principle" refers, here, to the use of the conventional potential function Π_1 and some modified terms for the construction of a special variational principle.

The proof of the principle and how to transform the modified functional into a boundary integral equation has been discussed in the above-mentioned paper (Jin and Qin, 1993). It will not, therefore, be repeated in detail here. In the case under consideration, the resulting formulation can be expressed in the form:

$$\alpha(P) u_k(P) - \int_{C_{N_n}} \{\overset{*}{u}{}_n^{(k)}(P,Q) \bar{N}_n - u_n \overset{*}{N}{}_n^{(k)}(P,Q)\} \, dc - \int_{C_{N_{ns}}} \{\overset{*}{u}{}_s^{(k)}(P,Q) \bar{N}_{ns} - u_s \overset{*}{N}{}_{ns}^{(k)}(P,Q)\} \, dc - \int_{C_{u_n}} \{\overset{*}{u}{}_n^{(k)}(P,Q) N_n - \bar{u}_n \overset{*}{N}{}_n^{(k)}(P,Q)\} \, dc - \int_{C_{u_s}} \{\overset{*}{u}{}_s^{(k)}(P,Q) N_{ns} - \bar{u}_s \overset{*}{N}{}_{ns}^{(k)}(P,Q)\} \, dc = -\int_\Omega \tfrac{1}{2} w_{,i} w_{,j} \overset{*}{N}{}_{ij}^{(k)}(P,Q) \, d\Omega \tag{27}$$

$$\alpha(P)u_m(P)-\int_{C_{M_n}}\{\overset{*}{\psi}{}_n^{(m)}(P,Q)\bar{M}_n-\psi_n\overset{*}{M}{}_n^{(m)}(P,Q)\}\,\mathrm{d}c$$

$$-\int_{C_{M_n}}\{\overset{*}{\psi}{}_s^{(m)}(P,Q)\bar{M}_{ns}-\psi_s\overset{*}{M}{}_{ns}^{(m)}(P,Q)\}\,\mathrm{d}c-\int_{C_R}\{\overset{*}{w}{}^{(m)}(P,Q)\bar{R}_n-w\overset{*}{Q}{}_n^{(m)}(P,Q)\}\,\mathrm{d}c$$

$$-\int_{C_{\psi_n}}\{\overset{*}{\psi}{}_n^{(m)}(P,Q)M_n-\bar{\psi}_n\overset{*}{M}{}_n^{(m)}(P,Q)\}\,\mathrm{d}c-\int_{C_{\psi_s}}\{\overset{*}{\psi}{}_s^{(m)}(P,Q)M_{ns}-\bar{\psi}_s\overset{*}{M}{}_{ns}^{(m)}(P,Q)\}\,\mathrm{d}c$$

$$-\int_{C_w}\{\overset{*}{w}{}^{(m)}(P,Q)R_n-\bar{w}\overset{*}{Q}{}_n^{(m)}(P,Q)\}\,\mathrm{d}c=\int_{\Omega}\{\overset{*}{w}{}^{(m)}(P,Q)q-N_{ij}w_{,i}\overset{*}{w}{}_{,j}^{(m)}(P,Q)\}\,\mathrm{d}\Omega,\tag{28}$$

where $\alpha(P)$ is a conventional boundary shape coefficient, $\alpha(P)=1$ if $P\in\Omega$, $\alpha(P)=1/2$ if P is on the smooth boundary $\partial\Omega$, $\{u_1\ \ u_2\ \ u_3\ \ u_4\ \ u_5\}=\{u\}=\{u_1\ \ u_2\ \ \psi_1\ \ \psi_2\ \ w\}$ is a displacement vector, and the asterisked symbol $(*)^{(p)}$ represents the related fundamental solution which has been obtained in subsection 2.2, the components $\overset{*}{u}{}_q^{(p)}(P,Q)$ of $\{\overset{*}{u}\}^{(p)}(P,Q)$ mean the in-plane displacements (for $q=1$ and 2) or the rotations (for $q=3$ and 4) or the deflection (for $q=5$) at the field point Q of an infinite plate when a unit point force (for $p=1$, 2 and 5) or a unit point couple (for $p=3$ and 4) is applied at the source point P. $\{\overset{*}{N}\}^{(p)}(P,Q)$ can be calculated from $\{\overset{*}{u}\}^{(p)}(P,Q)$ by using the constitutive relationships (2). All of the fundamental solutions, $\{\overset{*}{u}\}^{(p)}(P,Q)$, are given in the Appendix.

4. NUMERICAL IMPLEMENTATION

The analytical solutions of eqns (27) and (28) are not, in general, possible and therefore a numerical procedure must be used to solve the equations.

To obtain a numerical solution of eqns (27) and (28), as in the usual BEM the boundary $\partial\Omega$ and the domain Ω of the plate are divided into a series of constant boundary elements and constant cells, respectively. The node of an element is taken to be the centre of the element. After performing the discretization and introducing boundary conditions, eqns (27) and (28) are reduced to a system of algebraic equations:

$$[Q]\{N\}+[S]\{u\}=\{R_1\}+\{R_3\},\tag{29}$$

$$[H]\{M\}+[G]\{\psi\}=\{R_2\}+\{R_4\},\tag{30}$$

where $[Q]$, $[S]$, $[H]$ and $[G]$ denote the coefficient matrices which can be calculated in the usual way, $\{N\}=\{N_n\ \ N_{ns}\}$, $\{u\}=\{u_n\ \ u_s\}$, $\{M\}=\{R_n\ \ M_n\ \ M_{ns}\}$, $\{\psi\}=\{w\ \ \psi_n\ \ \psi_s\}$. These four vectors only include boundary variables, while $\{R_1\}$ and $\{R_2\}$ are inhomogeneous terms which can be deduced from (27) and (28), and $\{R_3\}$ and $\{R_4\}$ contain nonlinear terms:

$$\{R_3\}_q=-\int_{\Omega}\tfrac{1}{2}w_{,i}w_{,j}\overset{*}{N}{}_{ij}^{(k)}(P_q,Q)\,\mathrm{d}\Omega\tag{31a}$$

$$\{R_4\}_q=-\int_{\Omega}N_{ij}w_{,i}\overset{*}{w}{}_{,j}^{(m)}(P_q,Q)\,\mathrm{d}\Omega.\tag{31b}$$

Note that eqns (29) and (30) are not, in general, suitable for numerical analysis, an incremental form of the equations must therefore be adopted. Denoting the incremental variable by the superimposed dot and omitting the infinitesimal element resulting from the product of incremental variables, one obtains

$$[Q]\{\dot{N}\}+[S]\{\dot{u}\} = \{\dot{R}_1\}+\{\dot{R}_3\}, \tag{32a}$$

$$[H]\{\dot{M}\}+[G]\{\dot{\psi}\} = \{\dot{R}_2\}+\{\dot{R}_4\}. \tag{32b}$$

It follows that eqns (32a, b) are linear with respect to the incremental variables. However, the right-hand side vectors $\{\dot{R}_3\}$ and $\{\dot{R}_4\}$ contain domain unknown variables; to avoid solving these variables directly, an iterative procedure is required.

It is noted that $\{\dot{R}_3\}$ depends only upon $\dot{w}$ [see eqn (31a)]. So as long as the value of $\dot{w}$ in eqn (32b) has to be solved, we can compute the pseudo loading vector $\{\dot{R}_3\}$, and then, all of the unknown variables in eqn (32a) are at the boundary. We may solve it for $\dot{u}_1$ and $\dot{u}_2$. As a consequence, $\{\dot{R}_4\}$ can be evaluated from the current values of $\dot{u}_1$, $\dot{u}_2$ and $\dot{w}$. An iterative scheme may be established according to the above analysis. The scheme is quite similar to that in the paper of Qin and Huang (1990).

It is important to note that once the matrices $[Q]$, $[S]$, $[H]$ and $[G]$ in eqns (32a, b) have been formed, they can be stored in the core and used in each cycle of iteration without any change. That is because these matrices depend only upon the geometric and material parameters of plates and foundations. Obviously it can save a large amount of computing time.

5. NUMERICAL EXAMPLES

As numerical illustrations of the proposed method, three benchmark problems are considered. In order to allow for comparisons with other solutions appearing in the references (Azizian and Dawe, 1985; Katsikadelis, 1991; Ng and Das, 1986), the obtained numerical results are limited to a moderately thick plate with $k = 0$; a circular plate and a 60° skew sandwich plate resting on Winkler-type elastic foundations. To study the convergence properties of the proposed method, three meshes of the internal cell (or boundary element) are used in the analysis. The convergence tolerance is 0.001. These examples are described as following:

Example 1

A square plate with two opposite edges clamped and the others simply supported under a uniformly lateral load $q(Q = qa^4/Eh^4)$ and with thickness/span ratio $t/a = 0.05$. The boundary conditions are

$$x = \pm a/2, \quad u_n = u_s = w = \psi_n = \psi_s = 0,$$

$$y = \pm a/2, \quad u_n = u_s = w = M_n = \psi_s = 0.$$

Owing to the symmetry of the problem only one quadrant of the plate is modelled by 8 constant boundary elements and three meshes of the internal cell (3×3, 4×4 and 5×5). Table 1 shows the central deflection ($W_{\max}/t$) of the plate and compares the result with the finite strip solution (Azizian and Dawe, 1985).

Table 1. The centre deflection ($w_{\max}/t$) of the square plate

Load Q		0.91575	4.5788	6.8681	9.1575
Finite strip		0.019908	0.098873	0.14694	0.19361
BEM	16 cells	0.019903	0.098511	0.14571	0.19035
	25	0.019904	0.098623	0.14592	0.19127
	36	0.019907	0.098625	0.14598	0.19135

Table 2. Deflection $\bar{w}$ along the radius in a circular plate on an elastic foundation (load step $\Delta\bar{q} = 1$)

	r/a	0.098	0.304	0.562	0.800	0.960
Katsikadelis		1.108	0.961	0.592	0.179	0.009
Present	12 b.e.†	1.096	0.950	0.584	0.171	0.0085
	20	1.102	0.957	0.588	0.174	0.0088
BEM	30	1.109	0.960	0.590	0.175	0.0088

† b.e. = boundary elements.

Table 3. Variation of (w/h) with load Q for the sandwich plate ($K = 20$)

Q		25	50	75	100	125
BEM	48 b.e.†	0.581	0.864	1.0395	1.1692	1.2723
	60	0.589	0.867	1.0412	1.1723	1.2802
	80	0.596	0.872	1.0521	1.1750	1.2837
Exact†		0.60	0.87	1.05	1.18	1.30

† Values obtained from Ng and Das (1986), Fig. 11, p. 375.

Example 2

A uniformly loaded circular plate with radius R and clamped moveable edge (i.e. $w = \psi_n = \psi_s = N_n = N_{ns} = 0$) resting on an elastic foundation. Some parameters of the problem are assumed as

$$a/h = 50, \quad v = 0.3, \quad ka^4/D = 100, \quad Q = qa^4/Eh^4 = 15, \quad \bar{w} = w/h.$$

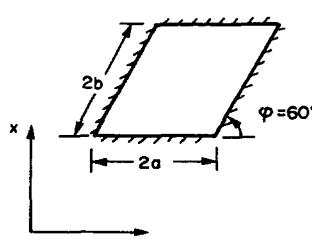
A quadrant of the plate is modelled by 25 internal cells and three meshes of boundary element (16, 20, 24). Some results obtained by the proposed method are listed in Table 2, and comparison is made with the known ones (Katsikadelis, 1991).

Example 3

Consider a uniformly loaded 60° skew sandwich plate which is clamped immovable (CI) on all edges (i.e. $u_1 = u_2 = \psi_1 = \psi_2 = w = 0$ on the whole boundary) and shown in Fig. 1.

The plate under consideration is modelled by 8×8 internal cells, three boundary meshes and (48, 60, 80) boundary elements, respectively. Some initial data are shown in Fig. 1. Table 3 compares the results obtained using the present BEM and the method given by Ng and Das (1986).

It can be seen from the three tables that the results are in excellent agreement with other solutions. In the course of the computations, convergence was achieved with about 25 iterations for example 1, 50 iterations for example 2 and 45 iterations for example 3 at each loading step. As expected for all three examples, it was found from the numerical



$v = 0.32$, $t = 0.635$ mm,
$h = 25.4$ mm, $a = b = 508$ mm,
$G_c = 6.89$ MPa, $z = t+h$,
$Q = 12a^3(1-v^2)q/(tz^2E)$,
$K = 12a^3k(1-v^2)/(ztE)$.

Fig. 1. A CI 60° skew sandwich plate resting on an elastic foundation.

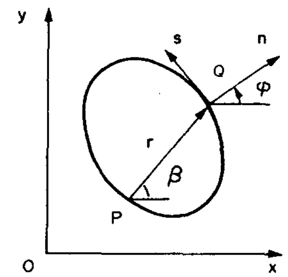


Fig. 2. The definition of β, ϕ and r.

results that the deflection converges gradually to the exact one along with refinement of the element meshes.

6. CONCLUDING REMARKS

Based on the procedure developed in the paper (Jin and Qin, 1993), a set of boundary integrals for a nonlinear Reissner plate resting on an elastic foundation is obtained. Another purpose of the paper is to obtain fundamental solutions for bending problems of Reissner plates on an elastic foundation by means of two of Hu's functions and resolution method of differential operator. Three numerical examples have been considered and the convergence is achieved with a relatively small number of boundary elements and iterative cycles. Although the proposed boundary integral equation and the numerical examples are confined to plates on Winkler-type foundation, it is easy to extend the procedure to the case of Pasternak-type foundations.

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APPENDIX: FUNDAMENTAL SOLUTIONS $\{\overset{*}{u}\}^{(m)}$

$$\overset{*}{u}{}_3^{(3)} = \psi_{n1}^* = [C(Z_3)+E(Z_1,Z_2)]\cos\beta\cos(\beta-\phi)+[D(Z_3)-F(Z_1,Z_2)]\cos(2\beta-\phi),$$

$$\overset{*}{u}{}_4^{(3)} = \psi_{s1}^* = [C(Z_3)+E(Z_1,Z_2)]\cos\beta\sin(\beta-\phi)-[D(Z_3)+F(Z_1,Z_2)]\sin(2\beta-\phi),$$

$$\overset{*}{u}{}_5^{(3)} = w_1^* = [A\sqrt{C_2}K_1(Z_1)(1-DC_2/C)+B\sqrt{C_1}Y_1(Z_1)(1-DC_2/C)]\cos(\beta),$$

$$\overset{*}{u}{}_3^{(4)} = \psi_{n2}^* = [C(Z_3)+E(Z_1,Z_2)]\sin\beta\cos(\beta-\phi)+[D(Z_3)-F(Z_1,Z_2)]\sin(2\beta-\phi),$$

$$\overset{*}{u}{}_4^{(4)} = \psi_{s2}^* = [C(Z_3)+E(Z_1,Z_2)]\sin\beta\sin(\beta-\phi)+[D(Z_3)+F(Z_1,Z_2)]\cos(2\beta-\phi),$$

$$\overset{*}{u}{}_5^{(4)} = w_2^* = [A\sqrt{C_2}K_1(Z_1)(1-DC_2/C)+B\sqrt{C_1}Y_1(Z_1)(1-DC_2/C)]\sin(\beta),$$

$$\overset{*}{u}{}_3^{(5)} = \psi_{n3}^* = -[B\sqrt{C_1}Y_1(Z_1)+A\sqrt{C_2}K_1(Z_2)]\cos(\beta-\phi),$$

$$\overset{*}{u}{}_4^{(5)} = \psi_{s3}^* = -[B\sqrt{C_1}Y_1(Z_1)+A\sqrt{C_2}K_1(Z_2)]\sin(\beta-\phi),$$

$$\overset{*}{u}{}_5^{(5)} = w_3^* = AC_2K_0(Z_2)[1-DC_2/C]+BY_0(Z_1)[1+DC_1/C],$$

where

$$A = \frac{1}{2\pi D(C_1+C_2)},\quad B = \frac{-1}{4D(C_1+C_2)},$$

$$C(Z_3) = \frac{-1}{(1-\nu)\pi D}\left[K_0(Z_3)+\frac{2}{Z_3}\left(K_1(Z_3)-\frac{1}{Z_3}\right)\right],$$

$$D(Z_3) = \frac{1}{(1-\nu)\pi D}\left[K_0(Z_3)+\frac{1}{Z_3}\left(K_1(Z_3)-\frac{1}{Z_3}\right)\right],$$

$$E(Z_1,Z_2) = BC_1Y_0(Z_1)-AC_2K_0(Z_2),$$

$$F(Z_1,Z_2) = BC_1Y_1(Z_1)/Z_1+AC_2K_1(Z_2)/Z_2,$$

$$Z_1 = \sqrt{C_1}r,\quad Z_2 = \sqrt{C_2}r,\quad Z_3 = \lambda r.$$